Supplement

$\mathbb{R}^4$-inv. instanton : $A = A_0 dx_0 + A_1 dx_1 + A_2 dx_2 + A_3 dx_3$

A_α: constant

$F_A = -*F_A \iff [A_0, A_1] + [A_2, A_3] = 0$ etc

<u>Nahm transform</u>

monopole $\longleftrightarrow$ Nahm's equation

more generally :

$\Lambda \subset \mathbb{R}^4$ subgroup

$\Lambda^* =$ dual of $\Lambda = \{ \lambda \in (\mathbb{R}^4)^* \mid \langle \lambda, \mu \rangle \in \mathbb{Z} \ \forall \mu \in \Lambda \}$

e.g. $\mathbb{R}^* = \mathbb{R}^3$ $\quad \{0\}^* = \mathbb{R}^4$

$\Lambda \subset \mathbb{R}^4$: lattice $\Rightarrow$ Λ^* : dual lattice

<u>Meta Theorem</u>

Λ-invariant instantons up to gauge equiv. $\overset{1:1}{\longleftrightarrow}$ Λ^*-invariant instantons up to gauge equiv.

- This statement is not precise unless boundary conditions are specified. But there are lots of examples of (Λ, Λ^*) s.t. the above holds (possibly after modifications e.g. allow singularities)

Ex, ① Λ : lattice $\quad \mathbb{R}^4/\Lambda = T^4_\Lambda$: torus

$\mathbb{R}^4/\Lambda^* =$ dual torus

Ok for irreducible instantons

② $\Lambda = \{0\}$, $\Lambda^* = \mathbb{R}^4$ $\longrightarrow$ ADHM transform

$\Lambda = \{0\}$ .... curvature $\in L^2$

$\Lambda^* = \mathbb{R}^4$ .... need "boundary correction terms"

V : hermitian vector space

$A = (A_0, A_1, A_2, A_3) \in \mathbb{R}^4 \otimes \mathfrak{u}(V)$ ... $\mathbb{R}^4$-invariant connection

W : another hermitian vector space

$\Phi \in S^+ \otimes \mathrm{Hom}(V, W)$ $\quad$ S^+ : +ve spinor $\quad \mathbb{C}^2$

$$\Phi^*\Phi \in \mathrm{End}(S^+) \otimes \mathrm{End}(V) \underset{\text{proj.}}{\rightsquigarrow} \underbrace{\mathfrak{su}(S^+)}_{\substack{\shortparallel \\ \Lambda^+}} \otimes \mathfrak{u}(V)$$

ADHM equation : $\underset{\text{etc}}{[A_0, A_1] + [A_2, A_3]} + \{\Phi^*, \Phi\} = 0$

— See Donaldson-Kronheimer for detail

③ $\Lambda = \mathbb{R}$, $\Lambda^* = \mathbb{R}^3$

monopole $\quad$ Nahm's equation

Th. (Hitchin, N (different proof)) $\qquad \Phi \sim i \begin{bmatrix} 1 - \frac{k}{2r} & 0 \\ 0 & -(1 - \frac{k}{2r}) \end{bmatrix}$

SU(2)-monopole with charge $k \in \mathbb{Z}_{>0}$

up to gauge

$\longleftrightarrow$ sol. of Nahm's equation on $(-1, 1) \times \mathbb{C}^k$

having 1st order pole at $t = \pm 1$

T_0 : regular $\quad T_\alpha \sim \dfrac{a_\alpha}{t \pm 1}$ + regular

$[a_\alpha, a_\beta] = \varepsilon_{\alpha\beta\gamma} a_\gamma$ --- rep. of $\mathfrak{su}(2)$

◎ It defines k-dim. irr. rep. of $\mathfrak{su}(2)$.

Suppose Λ: lattice. Choose a cpx structure on $\mathbb{R}^4$

$$\begin{tikzcd}
\text{instantons on } \mathbb{R}^4/\Lambda \arrow[r, "\text{Nahm transform}"] \arrow[d] & \text{instantons on } \mathbb{R}^4/\Lambda^* \arrow[d] \\
\text{holo. vector b'dle on } \mathbb{C}^2/\Lambda \arrow[r, "\text{Fourier-Mukai transform}"] & \text{holo. vector b'dles on } \mathbb{C}^2/\Lambda^*
\end{tikzcd}$$

This diagram commutes.

In fact, one can prove Nahm transf. by observing FM transf. can be defined independent of the choice of cpx str:

Recall FM transform

- Poincaré bundle

$(\mathbb{C}^2)^*/\Lambda^*$ = space of flat connections on $\mathbb{C}^2/\Lambda$

$\xi \in (\mathbb{C}^2)^* \qquad \Lambda \xrightarrow{\chi_\xi} U(1) \qquad \chi_\xi(\lambda) = e^{2\pi i \langle \lambda, \xi \rangle}$

$L_\xi = \mathbb{C}^2 \times \mathbb{C} / \Lambda \quad ; \quad (x, v) \sim (x+\lambda, \chi_\xi(\lambda) v)$

χ_ξ : trivial $\Leftrightarrow \xi \in \Lambda^*$ $\qquad$ connection form $\cdots 2\pi i \xi$ (const)

We can move ξ to make $x \leftrightarrow \xi$ symmetric

$$P = \mathbb{C}^2 \times (\mathbb{C}^2)^* \times \mathbb{C} / \Lambda \times \Lambda^* \quad ; \quad (x, \xi, v) \sim (x+\lambda, \xi + \eta, e^{2\pi i \langle \lambda, \xi \rangle + \langle x, \eta \rangle} v)$$

This has the universal connection: $A = 2\pi i(\underset{\substack{\uparrow \\ \text{constant 1-form on } \mathbb{C}^2}}{\xi} + \underset{\substack{\uparrow \\ (\mathbb{C}^2)^*}}{x})$

Now suppose E : holo. vector bdle on $\mathbb{C}^2/\Lambda$

Assume $H^i(E \otimes L_\xi)$ $i = 0, 2$ ⋯ vanish

$\hat{E}_\xi := H^1(E \otimes L_\xi)$: depending holomorphically on ξ

$\Longrightarrow$ $\hat{E}_\xi$: holo. vector bdle on $\mathbb{C}^2/\Lambda^*$

naive idea : $H^1(E \otimes L_\xi) = \mathrm{Ker}(\bar{\partial}_{A_\xi}) / \mathrm{Im}\, \bar{\partial}_{A_\xi}$

$$\Omega^{0,0} \xrightarrow[\bar{\partial}_{A_\xi}]{} \Omega^{0,1} \xrightarrow[\bar{\partial}_{A_\xi}]{} \Omega^{0,2}$$

depend holomorphically in ξ

◎ strategy for FM $\longrightarrow$ Nahm

Use Dirac operators instead of Dolbeault operators

defined independent of the choice of a cpx str.

$$\not{D}_{A_\xi} = \sqrt{2}\,(\bar{\partial}_{A_\xi} + \bar{\partial}^*_{A_\xi})$$

Define $\hat{E}_\xi = \mathrm{Ker}\, \not{D}^-_{A_\xi} \cong H^1(E \otimes L_\xi)$ by Hodge-Kodaira

has a metric & connection induced from $L^2(E \otimes L_\xi \otimes S^-)$

$\cong$ is compatible with the holo. structure

It is so for any choice of a cpx str.

$\Rightarrow$ instanton

Spinors & Dirac operators

$V = \mathbb{R}^d$, $(\ .\ )$: std metric

Clifford algebra : $Cl(V) \equiv Cl_d = T(V)/\sim$

$$x \otimes x \sim -(x.x)1$$

$d=4$

$$V = \mathbb{H} \qquad (x.y) = \mathrm{Re}(x\bar{y}) \qquad \bar{y} = y_0 - iy_1 - jy_2 - ky_3$$

For $x \in \mathbb{H}$ $\quad \begin{bmatrix} 0 & -x^* \\ x & 0 \end{bmatrix} \in M_2(\mathbb{H}) \qquad x^* = \bar{x}$

$$\begin{bmatrix} 0 & -x^* \\ x & 0 \end{bmatrix}\begin{bmatrix} 0 & -x^* \\ x & 0 \end{bmatrix} = -|x|_x^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad Cl_4 \cong M_2(\mathbb{H}) \curvearrowright S := \mathbb{H}^2 \quad \text{spinor repr.}$$

volume form : $e_0 e_1 e_2 e_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\Rightarrow S = \underset{\begin{bmatrix} 0 \\ * \end{bmatrix}}{S^+} \oplus \underset{\begin{bmatrix} * \\ 0 \end{bmatrix}}{S^-} \qquad \text{eigenspace decomposition}$$

$$\underset{\mathbb{H}\ni}{x\cdot} : S^{\pm} \longrightarrow S^{\mp}$$

Dirac operator $\quad D\!\!\!/_A^{\pm} := \sum_\alpha e_\alpha \cdot \nabla_{e_\alpha} : C^\infty(S^{\pm} \otimes E) \to C^\infty(S^{\mp} \otimes E)$

<u>Exercise</u>. $\quad D\!\!\!/_A^- D\!\!\!/_A^+ = -1_{S^+} \otimes \underbrace{\Delta_A}_{= \sum_\alpha \nabla_{e_\alpha}\nabla_{e_\alpha}} - F_A^+$

$\uparrow$ self-dual part

<u>Cor</u>. a) $F_A^+ = 0 \quad \Rightarrow \quad D\!\!\!/_A^- D\!\!\!/_A^+ = -1_{S^+} \otimes \Delta_A$

b) $F_A^+ = 0$ & A : irreducible $\Rightarrow \mathrm{Ker}\, D\!\!\!/_A^+ = \{0\}$

$\mathbb{H} \cong \mathbb{C} \oplus j\mathbb{C}$ mult. by $1, i, j, k$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

$$\not{D}_A^{\pm} = 1 \cdot \frac{\partial}{\partial x_0} + i \cdot \frac{\partial}{\partial x_1} + j \cdot \frac{\partial}{\partial x_2} + k \cdot \frac{\partial}{\partial x_3}$$

$$\rightsquigarrow \not{D}_A^- = 2\begin{bmatrix} \partial_{\bar z} & -\partial_{\bar w} \\ \partial_w & \partial_z \end{bmatrix}, \quad \not{D}_A^+ = 2\begin{bmatrix} -\partial_z & -\bar\partial_w \\ \partial_w & -\bar\partial_z \end{bmatrix}$$

$$(\bar\partial + \bar\partial^*)(f + g\, d\bar z \wedge d\bar w) = (\bar\partial_z f - \partial_w g)\, dz + (\bar\partial_w f + \partial_z g)\, dw$$

$d = 3$

$V = \mathbb{R}^3 \cong \operatorname{Im} \mathbb{H}$ $\quad \bar x = -x \quad \therefore \; x^2 = -x\bar x$

$= -|x|^2$

$Cl_3 \cong \mathbb{H} \oplus \mathbb{H}$

$\cup \qquad \cup$

$\mathbb{R}^3 \ni x \mapsto x \oplus -x \qquad (x \oplus -x)^2 = -|x|^2 (1 \oplus 1)$

$S = \mathbb{H}$: spinor b'dle

Dirac operator $\not{D}_A : C^\infty(S \otimes E) \longrightarrow C^\infty(S \otimes E)$

○ Nahm $\longrightarrow$ monopole $\qquad T_0(t), T_1(t), \cdots$: connection on $I \times V$ $(H.1)$

$x \in \mathbb{R}^3 \qquad S = \mathbb{C}^2$: spinor for $\mathbb{R}^3$

(x_1, x_2, x_3)

$$\not{D}_x^+ := 1_S \otimes \nabla_t + \sum (e_\alpha \otimes T_\alpha - i x_\alpha e_\alpha) \otimes 1_V$$

$\not{D}_x^-$:= its formal adjoint

- $\operatorname{Ker} \not{D}_x^+ = \{0\}$
- $E_x = \operatorname{Ker} \not{D}_x^- \subset L^2(I : V)$: vector bdle over $\mathbb{R}^3$

p : L^2-projection

Define
- connection $\nabla_{\frac{\partial}{\partial x_\alpha}} = p \circ \frac{\partial}{\partial x_\alpha}$
- Φ : Higgs field $= p \circ i t$ (t: time variable)

$\Rightarrow$ Bogomolny equ. is satisfied

○ monopole $\Rightarrow$ Nahm

$t \in I$

$$\not{D}_{A,t}^+ = \not{D}_A + (\Phi - it) \quad : C^\infty(S \otimes E) \to C^\infty(S \otimes E)$$

$$\not{D}_{A,t}^- = \not{D}_{A,t}^* = \not{D}_A - (\Phi - it)$$

$$\not{D}_{A,t}^- \not{D}_{A,t}^+ = 1_S \otimes \left(-\Delta_A - (\Phi - it)^2\right)$$

$$E_t = \operatorname{Ker} \not{D}_{A,t}^- \subset L^2$$

p : orthogonal proj.

- $\nabla_t = p \circ \frac{d}{dt}$
- $T_\alpha = p \circ x_\alpha \qquad \alpha = 1, 2, 3$

$\Rightarrow$ Nahm equ. is satisfied (& not so simple.

For the actual proof, study of bdry cond's is necessary)